Decoding Interleaved Gabidulin Codes using Alekhnovich's Algorithm

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Abstract

We prove that Alekhnovich's algorithm can be used for row reduction of skew polynomial matrices. This yields an $O(\ell^3 n^{(\omega+1)/2} \log(n))$ decoding algorithm for ℓ -Interleaved Gabidulin codes of length n, where ω is the matrix multiplication exponent, improving in the exponent of n compared to previous results.

Keywords: Gabidulin Codes, Characteristic Zero, Low-Rank Matrix Recovery

1 Introduction

It is shown in [1] that Interleaved Gabidulin codes of length $n \in \mathbb{N}$ and interleaving degree $\ell \in \mathbb{N}$ can be error- and erasure-decoded by transforming the

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following skew polynomial [2] matrix into weak Popov form (cf. Section 2)²:

$$\mathbf{B} = \begin{bmatrix} x^{\gamma_0} & s_1 x^{\gamma_1} & s_2 x^{\gamma_2} & \dots & s_\ell x^{\gamma_\ell} \\ 0 & g_1 x^{\gamma_1} & 0 & \dots & 0 \\ 0 & 0 & g_2 x^{\gamma_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & g_\ell x^{\gamma_\ell} \end{bmatrix},$$
(1)

where the skew polynomials $s_1, \ldots, s_\ell, g_1, \ldots, g_\ell$ and the non-negative integers $\gamma_0, \ldots, \gamma_\ell$ arise from the decoding problem and are known at the receiver. Due to lack of space, we cannot give a description of Interleaved Gabidulin codes, the mentioned procedure and the resulting decoding radius here and therefore refer to [1, Section 3.1.3]. By adapting row reduction³ algorithms known for polynomial rings $\mathbb{F}[x]$ to skew polynomials, a decoding complexity of $O(\ell n^2)$ can be achieved [1]. In this paper, we adapt Alekhnovich's algorithm [7] for row reduction of $\mathbb{F}[x]$ matrices to the skew polynomial case.

2 Preliminaries

Let \mathbb{F} be a finite field and σ an \mathbb{F} -automorphism. A skew polynomial ring $\mathbb{F}[x,\sigma]$ [2] contains polynomials of the form $a = \sum_{i=0}^{\deg a} a_i x^i$, where $a_i \in \mathbb{F}$ and $a_{\deg a} \neq 0$ (deg *a* is the *degree* of *a*), which are multiplied according to the rule $x \cdot a = \sigma(a) \cdot x$, extended recursively to arbitrary degrees. This ring is non-commutative in general. All polynomials in this paper are skew polynomials.

It was shown in [6] for linearized polynomials and generalized in [3] to arbitrary skew polynomials that two such polynomials of degrees $\leq s$ can be multiplied with complexity $\mathcal{M}(s) \in O(s^{(\omega+1)/2})$ in operations over \mathbb{F} , where ω is the matrix multiplication exponent.

A polynomial *a* has *length* len *a* if $a_i = 0$ for all $i = 0, \ldots, \deg a - \operatorname{len} a$ and $a_{\deg a - \operatorname{len} a + 1} \neq 0$. We can write $a = \tilde{a}x^{\deg a - \operatorname{len} a + 1}$, where $\deg \tilde{a} \leq \operatorname{len} a$, and multiply $a, b \in \mathbb{F}[x, \sigma]$ by $a \cdot b = [\tilde{a} \cdot \sigma^{\deg a - \operatorname{len} a + 1}(\tilde{b})]x^{\deg a + \deg a - \operatorname{len} b + 1}$. Computing $\sigma^i(\alpha)$ with $\alpha \in \mathbb{F}$, $i \in \mathbb{N}$ is in O(1) (cf. [3]). Hence, *a* and *b* of length *s* can be multiplied in $\mathcal{M}(s)$ time, although possibly deg *a*, deg $b \gg s$.

Vectors \mathbf{v} and matrices \mathbf{M} are denoted by bold and small/capital letters. Indices start at 1, e.g. $\mathbf{v} = (v_1, \ldots, v_r)$ for $r \in \mathbb{N}$. $\mathbf{E}_{i,j}$ is the matrix containing only one non-zero entry = 1 at position (i, j) and \mathbf{I} is the identity matrix. We denote the *i*th row of a matrix \mathbf{M} by \mathbf{m}_i . The degree of a vector $\mathbf{v} \in \mathbb{F}[x, \sigma]^r$ is the maximum of the degrees of its components deg $\mathbf{v} = \max_i \{ \deg v_i \}$ and

² Afterwards, the corresponding information words are obtained by ℓ many divisions of skew polynomials of degree O(n), which can be done in $O(\ell n^{(\omega+1)/2} \log(n))$ time [3].

 $^{^{3}}$ By row reduction we mean to transform a matrix into weak Popov form by row operations.

the degree of a matrix **M** is the sum of its rows' degrees deg $\mathbf{M} = \sum_{i} \deg \mathbf{m}_{i}$.

The leading position (LP) of \mathbf{v} is the rightmost position of maximal degree $LP(\mathbf{v}) = \max\{i : \deg v_i = \deg \mathbf{v}\}$. The leading coefficient (LC) of a polynomial a is $LT(a) = a_{\deg a} x^{\deg a}$ and the leading term (LT) of a vector \mathbf{v} is $LT(\mathbf{v}) = v_{LP(\mathbf{v})}$. A matrix $\mathbf{M} \in \mathbb{F}[x, \sigma]^{r \times r}$ is in weak Popov form (wPf) if the leading positions of its rows are pairwise distinct. E.g., the following matrix is in wPf since $LP(\mathbf{m}_1) = 2$ and $LP(\mathbf{m}_2) = 1$

$$\mathbf{M} = \begin{bmatrix} x^2 + x & x^2 + 1 \\ x^4 & x^3 + x^2 + x + 1 \end{bmatrix}.$$

Similar to [7], we define an accuracy approximation to depth $t \in \mathbb{N}_0$ of skew polynomials as $a|_t = \sum_{i=\deg a-t+1}^{\deg a} a_i x^i$. For vectors, it is defined as $\mathbf{v}|_t = (v_1|_{\min\{0,t-(\deg \mathbf{v}-\deg v_1)\}}, \dots, v_r|_{\min\{0,t-(\deg \mathbf{v}-\deg v_r)\}})$ and for matrices rowwise. E.g., with **M** as above,

$$\mathbf{M}|_2 = \begin{bmatrix} x^2 + x & x^2 \\ x^4 & x^3 \end{bmatrix} \text{ and } \mathbf{M}|_1 = \begin{bmatrix} x^2 & x^2 \\ x^4 & 0 \end{bmatrix}.$$

We can extend the definition of the length of a polynomial to vectors \mathbf{v} as $\operatorname{len} \mathbf{v} = \max_i \{ \operatorname{deg} \mathbf{v} - \operatorname{deg} v_i + \operatorname{len} v_i \}$ and to matrices as $\operatorname{len} \mathbf{M} = \max_i \{ \operatorname{len} \mathbf{m}_i \}$. With this notation, we have $\operatorname{len}(a|_t) \leq t$, $\operatorname{len}(\mathbf{v}|_t) \leq t$ and $\operatorname{len}(\mathbf{M}|_t) \leq t$.

3 Alekhnovich's Algorithm over Skew Polynomials

Alekhnovich's algorithm [7] was proposed for transforming matrices over ordinary polynomials $\mathbb{F}[x]$ into wPf. Here, we show that, with a few modifications, it also works with skew polynomials. As in the original paper, we prove the correctness of Algorithm 2 (main algorithm) using the auxiliary Algorithm 1.

Algorithm 1 R(M)

Input: Module basis $\mathbf{M} \in \mathbb{F}[x, \sigma]^{r \times r}$ with deg $\mathbf{M} = n$ Output: $\mathbf{U} \in \mathbb{F}[x, \sigma]^{r \times r}$: $\mathbf{U} \cdot \mathbf{M}$ is in wPf or deg $(\mathbf{U} \cdot \mathbf{M}) \leq \deg \mathbf{M} - 1$ 1. $\mathbf{U} \leftarrow \mathbf{I}$

- 2. While deg $\mathbf{M} = n$ and \mathbf{M} is not in wPf
- 3. Find i, j such that $LP(\mathbf{m}_i) = LP(\mathbf{m}_j)$ and $\deg \mathbf{m}_i \ge \deg \mathbf{m}_j$
- 4. $\delta \leftarrow \deg \mathbf{m}_i \deg \mathbf{m}_j \text{ and } \alpha \leftarrow \mathrm{LC}(\mathrm{LT}(\mathbf{m}_i))/\theta^{\delta}(\mathrm{LC}(\mathrm{LT}(\mathbf{m}_j)))$
- 5. $\mathbf{U} \leftarrow (\mathbf{I} \alpha x^{\delta} \mathbf{E}_{i,j}) \cdot \mathbf{U}$ and $\mathbf{M} \leftarrow (\mathbf{I} \alpha x^{\delta} \mathbf{E}_{i,j}) \cdot \mathbf{M}$
- $\boldsymbol{6}$. Return \mathbf{U}

Theorem 3.1 Algorithm 1 is correct and if $len(\mathbf{M}) \leq 1$, it is in $O(r^3)$.

Proof. Inside the while loop, the algorithm performs a so-called *simple trans*formation (ST). It is shown in [1] that such an ST on an $\mathbb{F}[x, \sigma]$ -matrix M preserves both its rank and row space (this does not trivially follow from the $\mathbb{F}[x]$ case due to non-commutativity) and reduces either $LP(\mathbf{m}_i)$ or deg \mathbf{m}_i . At some point, \mathbf{M} is in wPf, or deg \mathbf{m}_i and likewise deg \mathbf{M} is reduced by one. The matrix \mathbf{U} keeps track of the STs, i.e. multiplying \mathbf{M} by $(\mathbf{I} - \alpha x^{\delta} \mathbf{E}_{i,j})$ from the left is the same as applying an ST on \mathbf{M} . At termination, $\mathbf{M} = \mathbf{U} \cdot \mathbf{M}'$, where \mathbf{M}' is the input matrix of the algorithm. Since $\sum_i LP(\mathbf{m}_i)$ can be decreased at most r^2 times without changing deg \mathbf{M} , the algorithm performs at most r^2 STs. Multiplying $(\mathbf{I} - \alpha x^{\delta} \mathbf{E}_{i,j})$ by a matrix \mathbf{V} consists of scaling a row with αx^{δ} and adding it to another (target) row. Due to the accuracy approximation, all monomials of the non-zero polynomials in the scaled and the target row have the same power, implying a cost of r for each ST. The claim follows. \Box

We can decrease a matrix' degree by at least t or transform it into wPf by t recursive calls of Algorithm 1. We can write this as $R(\mathbf{M}, t) = \mathbf{U} \cdot R(\mathbf{U} \cdot \mathbf{M})$, where $\mathbf{U} = R(\mathbf{M}, t-1)$ for t > 1 and $\mathbf{U} = \mathbf{I}$ if t = 1. As in [7], we speed this method up by two modifications. The first one is a divide-&-conquer (D&C) trick, where instead of reducing the degree of a "(t-1)-reduced" matrix $\mathbf{U} \cdot \mathbf{M}$ by 1 as above, we reduce a "t'-reduced" matrix by another t-t' for an arbitrary t'. For $t' \approx t/2$, the recursion tree has a balanced workload.

Lemma 3.2 Let t' < t and $\mathbf{U} = \mathbf{R}(\mathbf{M}, t')$. Then, $\mathbf{R}(\mathbf{M}, t) = \mathbf{R}[\mathbf{U} \cdot \mathbf{M}, t - (\deg \mathbf{M} - \deg(\mathbf{U} \cdot \mathbf{M}))] \cdot \mathbf{U}.$

Proof. U reduces reduces deg M by at least t' or transforms M into wPf. Multiplication by $R[\mathbf{U} \cdot \mathbf{M}, t - (\deg \mathbf{M} - \deg(\mathbf{U} \cdot \mathbf{M}))]$ further reduces the degree of this matrix by $t - (\deg \mathbf{M} - \deg(\mathbf{U} \cdot \mathbf{M})) \ge t - t'$ (or $\mathbf{U} \cdot \mathbf{M}$ in wPf).

The second lemma allows to compute only on the top coefficients of the input matrix inside the divide-&-conquer tree, reducing the overall complexity.

Lemma 3.3 $R(M, t) = R(M|_t, t)$

Proof. Arguments completely analogous to the $\mathbb{F}[x]$ case of [7, Lemma 2.7] hold.

Lemma 3.4 $R(\mathbf{M}, t)$ contains polynomials of length $\leq t$.

Proof. The proof works as in the $\mathbb{F}[x]$ case, cf. [7, Lemma 2.8], by taking care of the fact that $\alpha x^a \cdot \beta x^b = \alpha \sigma^c(\beta) x^{a+b}$ for all $\alpha, \beta \in \mathbb{F}, a, b \in \mathbb{N}_0$. \Box

Algorithm 2 $\hat{\mathbf{R}}(\mathbf{M}, t)$ Input: Module basis $\mathbf{M} \in \mathbb{F}[x, \sigma]^{r \times r}$ with deg $\mathbf{M} = n$ Output: $\mathbf{U} \in \mathbb{F}[x, \sigma]^{r \times r}$: $\mathbf{U} \cdot \mathbf{M}$ is in wPf or deg $(\mathbf{U} \cdot \mathbf{M}) \leq \deg \mathbf{M} - t$

- 1. If t = 1, then Return $R(\mathbf{M}|_1)$
- 2. $\mathbf{U}_1 \leftarrow \hat{\mathbf{R}}(\mathbf{M}|_t, \lfloor t/2 \rfloor)$ and $\mathbf{M}_1 \leftarrow \mathbf{U}_1 \cdot \mathbf{M}|_t$
- 3. Return $\hat{\mathbf{R}}(\mathbf{M}_1, t (\deg \mathbf{M}|_t \deg \mathbf{M}_1)) \cdot \mathbf{U}_1$

Theorem 3.5 Algorithm 2 is correct and has complexity $O(r^3\mathcal{M}(t))$.

Proof. Correctness follows from $R(\mathbf{M}, t) = \hat{R}(\mathbf{M}, t)$ by induction (for t = 1, see Theorem 3.1). Let $\hat{\mathbf{U}} = \hat{R}(\mathbf{M}|_t, \lfloor \frac{t}{2} \rfloor)$ and $\mathbf{U} = R(\mathbf{M}|_t, \lfloor \frac{t}{2} \rfloor)$. Then,

$$\hat{\mathbf{R}}(\mathbf{M},t) = \hat{\mathbf{R}}(\hat{\mathbf{U}}\cdot\mathbf{M}|_{t}, t - (\deg\mathbf{M}|_{t} - \deg(\hat{\mathbf{U}}\cdot\mathbf{M}|_{t}))) \cdot \hat{\mathbf{U}}$$
$$\stackrel{(i)}{=} \mathbf{R}(\mathbf{U}\cdot\mathbf{M}|_{t}, t - (\deg\mathbf{M}|_{t} - \deg(\mathbf{U}\cdot\mathbf{M}|_{t}))) \cdot \mathbf{U} \stackrel{(ii)}{=} \mathbf{R}(\mathbf{M}|_{t}, t) \stackrel{(iii)}{=} \mathbf{R}(\mathbf{M}, t)$$

where (i) follows from the induction hypothesis, (ii) by Lemma 3.2, and (iii) by Lemma 3.3. Algorithm 2 calls itself twice on inputs of sizes $\approx \frac{t}{2}$. The only other costly operations are the matrix multiplications in Lines 2 and 3 of matrices containing only polynomials of length $\leq t$ (cf. Lemma 3.4). This costs r^4 r^2 times r multiplications $\mathcal{M}(t)$ and r^2 times r additions O(t) of polynomials of length $\leq t$, having complexity $O(r^3\mathcal{M}(t))$. The recursive complexity relation reads $f(t) = 2 \cdot f(\frac{t}{2}) + O(r^3\mathcal{M}(t))$. By the master theorem, we get $f(t) \in O(tf(1) + r^3\mathcal{M}(t))$. The base case operation $R(\mathbf{M}|_1)$ with cost f(1) is called at most t times since it decreases deg \mathbf{M} by 1 each time. Since $len(\mathbf{M}|_1) \leq 1$, $f(1) \in O(r^3)$ by Theorem 3.1. Hence, $f(t) \in O(r^3\mathcal{M}(t))$.

4 Implications and Conclusion

The orthogonality defect [1] of a square, full-rank, skew polynomial matrix \mathbf{M} is $\Delta(\mathbf{M}) = \deg \mathbf{M} - \deg \det \mathbf{M}$, where deg det is the "determinant degree" function, see [1]. A matrix \mathbf{M} in wPf has $\Delta(\mathbf{M}) = 0$ and deg det \mathbf{M} is invariant under row operations. Thus, if \mathbf{V} is in wPf and obtained from \mathbf{M} by simple transformations, then deg $\mathbf{V} = \Delta(\mathbf{V}) + \deg \det \mathbf{V} = \deg \mathbf{M} - \Delta(\mathbf{M})$. With $\Delta(\mathbf{M}) \geq 0$, this implies that $\hat{\mathbf{R}}(\mathbf{M}, \Delta(\mathbf{M})) \cdot \mathbf{M}$ is always in wPf. It was shown in [1] that \mathbf{B} from Equation (1) has orthogonality defect $\Delta(\mathbf{B}) \in O(n)$, which implies the following theorem.

Theorem 4.1 (Main Statement) $\hat{R}(\mathbf{B}, \Delta(\mathbf{B})) \cdot \mathbf{B}$ is in wPf. This implies that we can decode Interleaved Gabidulin codes in⁵ $O(\ell^3 n^{(\omega+1)/2} \log(n))$.

⁴ In D&C matrix multiplication algorithms, the length of polynomials in intermediate computations might be much larger than t. Thus, we have to compute it naively in cubic time. ⁵ The log(n) factor is due to the divisions in the decoding algorithm, following the row reduction step (see Footnote 2) and can be omitted if log(n) $\in o(\ell^2)$.

Table 1 compares the complexities of known decoding algorithms for Interleaved Gabidulin codes. Which algorithm is asymptotically fastest depends on the relative size of ℓ and n. Usually, one considers $n \gg \ell$, in which case the algorithms in this paper and in [4] provide—to the best of our knowledge—the fastest known algorithms for decoding Interleaved Gabidulin codes.

Algorithm	Complexity
Skew Berlekamp–Massey [5]	$O(\ell n^2)$
Skew Berlekamp–Massey (D&C) [4]	$O(\ell^K n^{\frac{\omega+1}{2}} \log(n)), \text{ possibly } {}^6 K = 3$
Skew Demand–Driven [*] [1]	$O(\ell n^2)$
Skew Alekhnovich [*] (Theorem 3.5)	$O(\ell^3 n^{\frac{\omega+1}{2}} \log(n)) \subseteq^{\dagger} O(\ell^3 n^{1.69} \log(n))$
Table 1	

Comparison of decoding algorithms for Interleaved Gabidulin codes. Algorithms marked with * are based on the row reduction problem of [1]. [†]Example $\omega \approx 2.37$.

In the case of Gabidulin codes $(\ell = 1)$, we obtain an alternative to the *Linearized Extended Euclidean* algorithm from [6] of the same complexity. The algorithms are equivalent up to the implementation of a simple transformation.

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⁶ In [4], the complexity is given as $O(n^{\frac{\omega+1}{2}}\log(n))$ and ℓ is considered to be constant. By a rough estimate, the complexity becomes $O(\ell^{O(1)}n^{\frac{\omega+1}{2}}\log(n))$ when including ℓ . We believe the exponent of ℓ is really 3 (or possibly ω) but this should be further analyzed.